

The Groups G_n^k and Fundamental Groups of Configuration Spaces

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Abstract

We construct a map from fundamental groups of complements to some plane configurations to the groups G_n^k for large k .

1 Introduction. Basic Notions

In the paper [3], the author introduced a family of groups G_n^k depending on two positive integers $n > k$, and formulated the following principle: *if a dynamical system describing a motion of n particles admits some nice general position codimension 1 property governed exactly by k particles, then this dynamical system has invariants valued in G_n^k .*

In [5], two partial cases were calculated explicitly. If we consider a motion of n pairwise distinct points on the plane and choose the property “some three points are collinear”, then we get a homomorphism from the pure n -strand braid group PB_n to the group G_n^3 . If we choose the property “some four points belong to the same circle or line”, we shall get a map from PB_n to G_n^4 . The approach with G_n^3 has been upgraded in [4].

In the present paper we construct maps from fundamental groups of configuration spaces to the groups G_n^k .

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We are interested in the configuration spaces $C_n(\mathbb{R}^{k-1})$ of ordered sets of n points in $(k-1)$ -dimensional space ($n > k$). For $k = 3$ the fundamental group of this space $\pi_1(C_n(\mathbb{R}^2))$ is precisely the Artin pure braid group. For $k > 3$ these spaces are simply connected.

Thus we shall consider spaces $C'_n(\mathbb{R}^{k-1})$ defined as follows: a point in $C'_n(\mathbb{R}^{k-1})$ is a set of n distinct points in \mathbb{R}^{k-1} , where every $(k-1)$ points are in general position. Thus, for example, for $k = 3$, the only condition is that no two points among the given n points coincide. In other words, $C'_n(\mathbb{R}^2) = C_n(\mathbb{R}^2)$. For $k = 4$, for points x_1, \dots, x_n in three-space we require that no three points are collinear (though some four points can belong to the same plane), for $k = 5$ a point in $C'_n(\mathbb{R}^4)$ is an n -tuple of points in \mathbb{R}^4 with no four of them belonging to the same 2-plane, etc. Quite analogously, one defines $C_n(\mathbb{R}P^{k-1})$ and $C'_n(\mathbb{R}P^{k-1})$.

Following [3], let us define the groups G_n^k . For positive integers $n > k$, the group G_n^k has $\binom{n}{k}$ generators a_m , where m runs all unordered sets of k distinct indices from 1 to n .

$$G_n^k = \langle a_m | (1), (2), (3) \rangle,$$

where (1) means that $(a_m)^2 = 1$ for each unordered set $m \subset \{1, \dots, n\}$, $\text{Card}(m) = k$; (2) means that $a_m a_{m'} = a_{m'} a_m$, whenever $\text{Card}(m \cap m') < k-1$; finally, (3) looks as follows. For every set $U \subset \{1, \dots, n\}$ of cardinality $k+1$, let us order arbitrarily all its k -subsets: m^1, \dots, m^{k+1} . Then $(a_{m^1} \cdots a_{m^{k+1}})^2 = 1$. This relation can also be rewritten as $a_{m^1} \cdots a_{m^{k+1}} = a_{m^{k+1}} \cdots a_{m^1}$.

Note that for $n = k+1$ the relations of type (2) never happen, since every two distinct subsets m, m' of cardinality k of $\{1, \dots, k+1\}$ have intersection of cardinality $k-1$.

Let us fix two positive integers $n > k$. Let us consider the space

$C'_n(\mathbb{R}^{k-1})$. A point $x \in C'_n(\mathbb{R}^{k-1})$ is *singular*, if the set of points $x = (x_1, \dots, x_n)$ representing it contains some k points which are not in general position, i.e., belong to the same $(k - 2)$ -plane. Let us fix two nonsingular points $x, x' \in C'_n(\mathbb{R}^{k-1})$.

Let us consider the set of smooth paths $\gamma_{x,x'} : [0, 1] \rightarrow C'_n(\mathbb{R}^{k-1})$. For every such path we have a set of values t for which the set of points corresponding to $\gamma_{x,x'}(t)$ is not in general position (some k points belong to the same $k - 2$ -plane). We call these values $t \in [0, 1]$ *singular*. We say that a smooth path is *good and stable* if the following conditions hold:

1. The set of singular values t is finite;
2. For each singular value $t = t_l$, among the n points representing $\gamma_{x,x'}(t_l)$ there exists exactly one subset of k points belonging to the same $(k - 2)$ -plane;
3. A smooth path is *stable*, i.e., the number of singular values does not change under a small perturbation.

We say that two paths γ, γ' with the same endpoints x, x' are *isotopic*, if there exists a continuous family $\gamma_{x,x'}^s, s \in [0, 1]$, of smooth paths with endpoints fixed, such that $\gamma_{x,x'}^0 = \gamma, \gamma_{x,x'}^1 = \gamma'$. A smooth path is a *braid*, if x and x' correspond to the same set of points (possibly, up to a permutation). A smooth path is a *pure braid* if x and x' coincide as ordered sets. We can make every smooth path good and stable with the the ends fixed by an arbitrarily small perturbation. A braid can also be *good and stable*; if it is not so, we can make it good and stable by an arbitrarily small perturbation.

We shall consider the set of smooth paths up to isotopy. For paths $\gamma_{x,x'}$ and $\gamma'_{x',x''}$ the *concatenation* operation is well defined; one should consider

the path $\gamma''_{x,x''}$, such that $\gamma''(t) = \gamma(2t)$ for $t \in [0, \frac{1}{2}]$ and $\gamma''(t) = \gamma'(2t-1)$ for $t \in [\frac{1}{2}, 1]$ and smooth it in the neighbourhood of $t = \frac{1}{2}$; this smoothing is unique up to isotopy.

Hence, the set of braids and the set of pure braids (for a fixed x) admits a group structure. The latter group is, obviously, isomorphic to the fundamental group of the configuration space $\pi_1(C'_n(\mathbb{R}^{k-1}))$. The former group is isomorphic to the fundamental group of the quotient space of this configuration space by the obvious action of the permutation group.

2 The Main Theorem

Now with each good and stable path we associate an element of the group G_n^k as follows. Let us enumerate all singular values $0 < t_1 < \dots < t_l < 1$ of our smooth path (it is agreed that 0 and 1 are non-singular). For each singular value t_p there exists exactly one unordered set m_p of k indices corresponding to this value. Let us associate a_{m_p} to it. With the whole path γ we associate the product $f(\gamma) = a_{m_1} \cdots a_{m_l}$.

Theorem 1. *The map f described above takes isotopic paths $\gamma_{x,x'}, \gamma'_{x,x'}$ with the endpoints fixed (x, x' are non-singular) to the same element of the group G_n^k . For pure braids, the map f is a homomorphism $f : \pi_1(C'_n(\mathbb{R}^{k-1})) \rightarrow G_n^k$.*

Proof. The proof follows from the basic principle from [3] that in order to consider isotopy between two paths, it suffices to take into account only singularities of codimension at most two (and not to count higher codimension singularities). As singularities of codimension one (k -tuples of points on the same $(k-2)$ -plane) give rise to generators, relators correspond to singularities of codimension two. Let us list them explicitly.

1. A non-stable set of k points on a $(k - 2)$ -plane which disappears after a small perturbation; this corresponds to the relation $a_m^2 = 1$;
2. Coincidence of two moments when two independent k -tuples appear. This corresponds to the relations $a_p a_q = a_q a_p$, where the sets p, q have no more than $k - 2$ common indices;
3. The situation when some $k + 1$ points belong to the same $(k - 2)$ -plane. This path is not good.

We can make it good by a small perturbation; the resulting good path will contain some $k + 1$ moments when each $k + 1$ subsets of k points belong to the same plane [3]. The “opposite” small perturbation leads to the opposite order of these $k + 1$ non-stable subsets, which follows from the general position argument from [3]. This leads to the relation (3).

□

Remark 1. In the case $k = 3$, the map described above coincides with the map from the pure braid group to G_n^3 , see [5].

The same arguments as above lead to the following

Theorem 2. *Theorem 1 holds if we replace \mathbb{R}^{k-1} with $\mathbb{R}P^{k-1}$.*

3 Hierarchy, Duality, and Open Questions

The groups G_n^k have a hierarchy $\cdots \subset G_n^k \subset G_{n+1}^{k+1} \subset G_{n+2}^{k+2} \subset \cdots$, where the monomorphism $G_n^k \rightarrow G_{n+1}^{k+1}$ is given by the formula $a_m \mapsto a_{m \cup \{n+1\}}$; the inverse map (projection) is given by the formula $a_m \mapsto 1$, if $(n + 1) \notin m$, $a_m \mapsto a_{m \setminus \{n+1\}}$, if $(n + 1) \in m$.

This hierarchy agrees with embeddings of (projective) spaces and the maps of fundamental groups to G_n^k described above, as follows.

With each dynamical system of n pairwise distinct points in $\mathbb{R}P^{k-1}$ we can associate a dynamical system of $n+1$ points in $\mathbb{R}P^k$ with homogeneous coordinates $(y_1 : y_2 : \dots : y_k : y_{k+1})$, such that the $(n+1)$ -th point is fixed with coordinates $(0 : 0 : \dots : 0 : 1)$, and j -th point $j = 1, \dots, n$, has the same coordinates as those of the initial dynamics, and the last coordinate is equal to some constant s_j . Whatever coordinate s_j we choose, any singular value of the initial dynamical system leads to a singular value for the resulting dynamical system: k points on a $(k-2)$ -dimensional plane together with the $(n+1)$ -th point will lie on a $(k-1)$ -plane.

Our goal is to choose all s_j in such a way that no other singular values occur for the new dynamics. This can be done by consequently choosing all s_j positive and large enough in comparison with all previous s_1, \dots, s_{j-1} . Namely, the condition that some $k+1$ points in $\mathbb{R}P^k$ lie on the same $(k-1)$ -plane means that some determinant equals zero. The determinant in question will be equal to the sum $D = \pm s_{m_1} D_1 \pm s_{m_2} D_2 \pm \dots \pm s_m D_m$ where D_j are some minors, which are all non-zero because no $(k-1)$ points belong to the same $(k-3)$ -plane. By using the compactness argument, all D_i attain their minimum and maximum absolute values, so we can choose all s_j such that the sign of D is the same as that of the “leading term” $\pm s_m D_m$ if s_m is large enough. Now, the compactness argument can be used not only for a single braid but also for an isotopy between two braids. In this case, the word corresponding to the new dynamics will be obtained from the word corresponding to the initial dynamics by replacing each a_m , ($m \subset \{1, \dots, n\}$) with $a_{m \cup \{(n+1)\}}$, hence the resulting word will be the image of the inclusion map $G_n^k \rightarrow G_{n+1}^{k+1}$.

Studying the image of the group $\pi_1(C'_n(\mathbb{R}^{k-1}))$ in the group G_n^k is quite an interesting problem. This map is neither injective nor surjective (we can see that it is non-injective even for $k = 3$: the pure braid corresponding to the cyclic rotation of the circle containing all n points, is mapped to the unit element of the group G_n^k because no three points belonging to the same circle are collinear). This map can be made injective if we consider a bit more complicated group than G_n^3 , see [6].

As for the image Imf , the following holds. Let us fix some $k-1$ indices $m_1, \dots, m_{k-1} \in \{1, \dots, n\}$, and order the remaining indices: l_1, \dots, l_{n-k+1} . We say that an element of G_n^k is *elementary* (or ϵ -type) element, if it is equal to $(a_{m_1, \dots, m_{k-1}, l_1} \cdots a_{m_1, \dots, m_{k-1}, l_{n-k+1}})^2$.

Theorem 3. *For $k > 3$ every element from $Imf \in G_n^k$ is a product of ϵ -type elements and their conjugates.*

We can sketch the proof as follows (we illustrate it for $k = 4$; for larger k it is analogous). Let some point x_3 lie in a small δ -neighbourhood of a line passing through the points x_1, x_2 in three-space. Let us rotate x_3 around this line. Then the plane passing through x_1, x_2, x_3 will rotate together with x_3 ; after the 2π -rotation, for each of the points $x_j, j \neq 1, 2, 3$, there will be two moments when x_j belongs to the plane passing through x_1, x_2, x_3 ; these two moments differ by a π -rotation. If we choose δ small enough, there will be no other singular moments; this means that we get exactly the square of some word of length $(n-3)$, i.e., an ϵ -type element. It is clear that instead of 1, 2, 3 we can consider any three indices (for $k = 4$).

The fact that the image of Imf consists only of products of conjugates to ϵ -type elements, can be seen as follows. The space $C_n(\mathbb{R}^{k-1})$ is simply connected. Hence, in order to understand the fundamental group of the

space $C'_n(\mathbb{R}^{k-1})$ one should see what happens in the neighbourhood of the complement $C_n(\mathbb{R}^{k-1}) \setminus C'_n(\mathbb{R}^{k-1})$, i.e., in the neighbourhood of sets of those points where some $k-1$ points belong to the same $(k-3)$ -plane. The elements of G_n^k corresponding to cycles around such deleted subsets are exactly ϵ -type elements. Thus, with a reference point A fixed, we can decompose each path in $C'_n(\mathbb{R}^{k-1})$ into a collection of paths of the following type. Each path approaches complement $C_n(\mathbb{R}^{k-1}) \setminus C'_n(\mathbb{R}^{k-1})$, rotates around this complement, and returns back to A . This precisely means that we have a product of conjugates of ϵ -type elements

Certainly, this argument fails for $\pi_1(C'_n(\mathbb{R}P^{k-1}))$ since real projective spaces are not simply connected.

According to the general principle of [3], the particle should not necessarily be a point. For example one can treat Theorems 1,2 by using projective duality.

Namely, let us consider the space of sets of n (projective) hyperplanes in k -dimensional real (projective) space, where every $k-1$ hyperplanes are in general position. The problem of studying of this space is exactly the problem of studying $C'_n(\mathbb{R}^k)$ (resp., $C'_n(\mathbb{R}P^k)$).

This naturally extends the application domain of groups G_n^k to the study of various subspaces of various codimensions.

It would be extremely interesting to establish the connection between G_n^k with the Manin-Schechtmann “higher braid groups” [8], where the authors study the fundamental group of complements to some configurations of *complex hyperplanes*. One of the relations in the paper of M.Kapranov and V.Voevodsky [7] looks quite similar to our relation (3) for G_n^k .

Besides the questions mentioned above, the following question seems quite interesting to the author: how to realize G_n^k for $k \geq 5$, by a motion

of points on \mathbb{R}^2 ; this will lead to a homomorphism of the pure braid group to $G_n^k, k \geq 5$. As mentioned above, there is a homomorphism from PB_n to G_n^4 .

It seems likely that the connections between the Artin braid group and all such geometric realizations of groups G_n^k can be achieved in the following manner. Take an open disc in \mathbb{R}^2 and embed it smoothly into \mathbb{R}^{k-1} in such a way that no $(k-1)$ points on the embedded surface belong to the same $(k-3)$ -plane. Then we obviously get a map from the pure braid group to G_n^k . For $k = 4$ we can map the disc $\{x, y | x^2 + y^2 < 1\}$ to the half-sphere: $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$ which would lead to a nice picture on the half-sphere: no three points on the half-sphere are collinear, and if four points on the sphere belong to the same plane, then they belong to the same circle, cf. with [5].

So, the following natural question arises: which topological spaces T can be (smoothly) embedded into \mathbb{R}^{k-1} in such a way that no $k-1$ points in the image of T belong to the same $(k-1)$ plane?

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References

- [1] V.G.Bardakov, The Virtual and Universal Braids, *Fundamenta Mathematicae*, 184 (2004), P. 1-18.
- [2] V.O.Manturov, On the Groups G_n^2 and Coxeter Groups, <http://arxiv.org/abs/1512.09273>.
- [3] V.O.Manturov, Non-Reidemeister Knot Theory and Its Applications in Dynamical Systems, Geometry, and Topology, <http://arxiv.org/abs/1501.05208>
- [4] V.O.Manturov, Invariants of Classical Braids Valued in G_n^2 , <http://arxiv.org/abs/1611.07434v2>.
- [5] V.O.Manturov, I.M.Nikonov, On braids and groups G_n^k J. Knot Theory & Ramifications, 2015, Vol. 24, No.13.
- [6] S.Kim, V.O.Manturov, The Braid Groups and Imaginary Generators, <http://arxiv.org/abs/1612.03486v1>.

- [7] M.Kapranov, V.Voevodsky, Braided Monoidal 2-Categories and Manin-Schechtmann Higher Braid Groups, Journal of Pure and Applied Mathematics, N. 92, 1994, P. 241-167.
- [8] Y.I.Manin, V.V.Schechtmann, Arrangements of Hyperplanes, higher braid groups and higher Bruhat orders, Adv. Stud. Pure Math., 17 (1990), P. 289-308